Lecture 03
12.3/12.4 Projections and the cross product

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Last Class

Definition

If $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$, $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, *and* $k \in \mathbb{R}$, *then we have the following operations:*

Vector addition:

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

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Scalar multiplication:

$$k\vec{\mathbf{u}} = \langle ku_1, ku_2, ku_3 \rangle.$$

Definition

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. *Then*

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

is the dot product of $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$.

Orthogonal Vectors

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Vectors $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *are* <u>*orthogonal*</u> *if* $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Orthogonal examples

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Example

Let $\vec{\mathbf{u}} = \langle 3, -2 \rangle$ *and* $\vec{\mathbf{v}} = \langle 4, 6 \rangle$. *Then*

Orthogonal examples

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Let $\vec{\mathbf{u}} = \langle 3, -2 \rangle$ *and* $\vec{\mathbf{v}} = \langle 4, 6 \rangle$. *Then* $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 3(4) + (-2)(6) = 0$.
So $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *are orthogonal.*

Orthogonal examples

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Example

Similarly, $\vec{\mathbf{0}}$ *and any other vector are orthogonal, since*
$\vec{\mathbf{0}} \cdot \vec{\mathbf{u}} = 0(u_1) + 0(u_2) + 0(u_3) = 0$.

Properties of the dot product

The dot product satisfies the following properties (page 721):

1. $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$
2. $(c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}}) = c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})$
3. $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
4. $\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} = \|\vec{\mathbf{u}}\|^2$
5. $\vec{\mathbf{0}} \cdot \vec{\mathbf{u}} = 0$

Properties of the dot product

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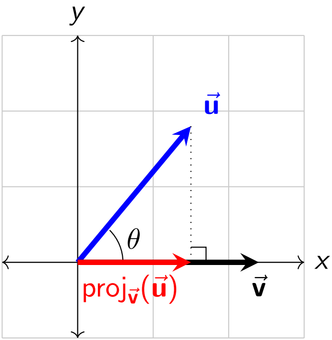
Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$.

$\vec{\mathbf{u}} \cdot \vec{\mathbf{u}}$	Left hand side of equation
$= u_1(u_1) + u_2(u_2) + u_3(u_3)$	Definition of dot product
$= u_1^2 + u_2^2 + u_3^2$	Simplified
$= \|\vec{\mathbf{u}}\|^2$	Definition of length

Vector projections

We can project one vector onto another vector.



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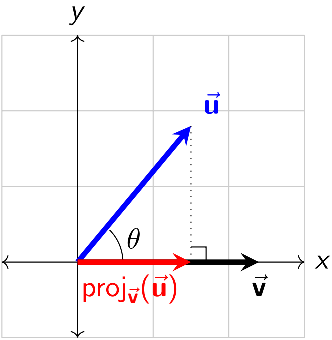

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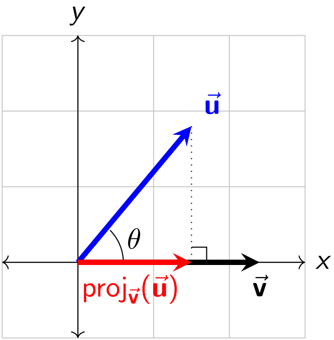

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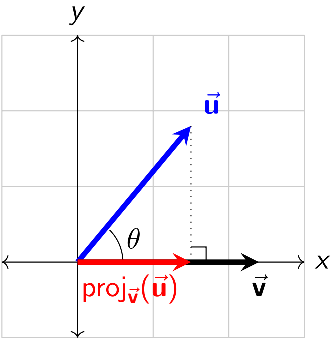

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$$= \left(\frac{\|\vec{\mathbf{u}}\|(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}\right)\left(\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}\right).$$

This can be cleaned up a bit.

Vector projections

$$\left(\frac{\|\vec{\mathbf{u}}\|(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})}{\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\|}\right)\left(\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}\right)$$

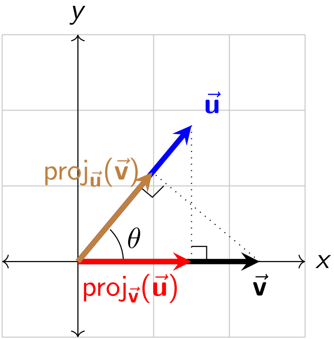
Definition

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be nonzero vectors. Then the projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$ is

$$proj_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \left(\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|^2}\right)\vec{\mathbf{v}}.$$

Projection properties

$$\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) \neq \text{proj}_{\vec{\mathbf{u}}}(\vec{\mathbf{v}})$$



Projection properties

The number $\|\vec{\mathbf{u}}\|\cos(\theta)$ $\left(= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}\right)$ is called the scalar component of $\vec{\mathbf{u}}$ in the direction of $\vec{\mathbf{v}}$.

Projection example

Example

Let $\vec{\mathbf{u}} = 6\vec{\mathbf{i}} + 3\vec{\mathbf{j}} + 2\vec{\mathbf{k}}$ *and* $\vec{\mathbf{v}} = \vec{\mathbf{i}} - 2\vec{\mathbf{j}} - 2\vec{\mathbf{k}}$.

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Let $\vec{\mathbf{u}} = 6\vec{\mathbf{i}} + 3\vec{\mathbf{j}} + 2\vec{\mathbf{k}}$ *and* $\vec{\mathbf{v}} = \vec{\mathbf{i}} - 2\vec{\mathbf{j}} - 2\vec{\mathbf{k}}$*. Then*

$$proj_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|^2}\right)\vec{\mathbf{v}} = \left(\frac{6 - 6 - 4}{1 + 4 + 4}\right)\langle 1, -2, -2\rangle$$

$$= -\frac{4}{9}\langle 1, -2, -2, \rangle = \langle -\frac{4}{9}, \frac{8}{9}, \frac{8}{9}\rangle.$$

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We want to define another vector product, the cross product, which we will denote as $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

Whereas the dot product gave us a number, we now want a vector product that gives us a vector.

To find a new vector, we need a length and a direction.

Direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$

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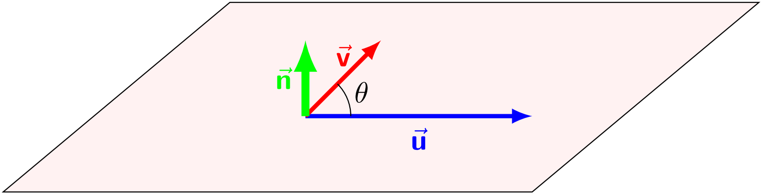
The direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is the unit vector $\vec{\mathbf{n}}$ shown in the picture below.



The direction of $\vec{\mathbf{n}}$ (up, rather than down), is chosen with the right-hand rule.

Length of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$

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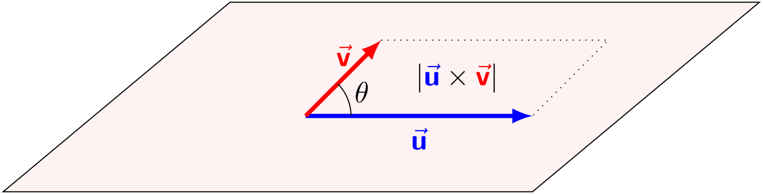
The length of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is the area of the parallelogram formed by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.



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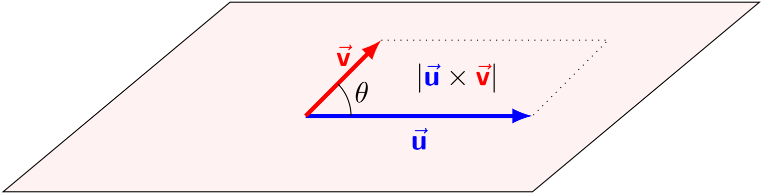

This area is $\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \sin(\theta)$.

The cross product

Definition

The cross product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, denoted $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, is the vector

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \sin(\theta))\vec{\mathbf{n}}.$$